ON THE OPTIMAL TIMING OF REGIME SWITCHING IN OPTIMAL GROWTH MODELS: A SOBOLEV SPACE APPROACH

A Master's Thesis

by

EROL DOĞAN

Department of

Economics

Bilkent University

Ankara

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by

EROL DOĞAN

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Hüseyin Çağrı Sağlam
Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Dr. Semih Koray
Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Hande Yaman
Examining Committee Member

Approval of the Institute of Economics and Social Sciences

Prof. Dr. Erdal Erel
Director

ABSTRACT

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Erol Doğan

M.A., Department of Economics

Supervisors: Assist. Prof. Dr. Hüseyin Çağrı Sağlam

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In this thesis, we derive the necessary conditions of optimality of regime switching in optimal growth models, and extend the already established results in the literature to cover the multi-stage infinite horizon models depending explicitly to the switching instant. To this end, we utilize standard techniques from calculus of variations, and some basic properties of Sobolev spaces. We compare our results with previously established ones. In an application, we analyze the determinants of timing of technological adoption, under linearly expanding technological frontier.

Keywords: Multi-stage Optimal Control, Optimal Growth, Adoption, Sobolev Spaces.

ÖZET

OPTİMAL BÜYÜME MODELLERİNDE REJİM DEĞİŞİKLİĞİNİN ZAMANLAMASI ÜZERİNE: SOBOLEV UZAYI YAKLAŞIMI

Erol Doğan

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Yrd. Doç. Dr. Hüseyin Çağrı Sağlam

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Bu tezde optimal büyüme modellerinde rejim değişikliğinin optimal oluşu için gerekli koşulları buluyor, ve literatürde varolan sonuçları, çok aşamalı, sonsuz ufuklu, değişiklik anına açıkça bağımlı modellere genişletiyoruz. Bu amaçla, varyasyonlar kalkülüsünden standart tekniklerle birlikte Sobolev uzaylarının temel özelliklerini kullanıyoruz. Sonuçlarımızı önceden bulunmuş sonuçlarla karşılaştırıyoruz. Bir uygulamada, doğrusal olarak genişleyen bir teknoloji cephesi altında teknolojik adaptasyonun zamanlamasını belirleyen faktörleri inceliyoruz.

Anahtar Kelimeler: Çok Aşamalı Optimal Kontrol, Optimal Büyüme, Adaptasyon, Sobolev Uzayları.

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CHAPTER 1

INTRODUCTION

Optimal growth models are useful tools to analyze dynamics of an economy. The role of technology in this dynamics is of particular importance. So comes the technology adoption problem to the stage, together with a set of questions related to the determinants of the timing of adoption. The effect of learning and maintenance on the adoption of the new technology are just two examples. As learning and maintenance may adversely affect the adoption of new technology by causing delays in the adoption, it is important to have a precise idea about the mechanics of such a delay (Boucekkine et al., 2004). Adoption under the conditions of continuously increasing technology frontier is also a problem of this kind. But one needs the tools to deal with these problems. Since these problems include a succession of different technological regimes, traditional dynamic optimization methods do not trivially extend to these problems. Below, we will be discussing the techniques associated with these problems. Also a model with increasing technology frontier will be analyzed so that we will be able to carry the discussion on the technique on material grounds, and highlight some aspects of technological adoption.

Maximization of a functional

$$\int_0^\infty U(c(t))e^{-\rho t}dt$$

subject to constraints of type

$$\begin{aligned}
\dot{k}(t) + q_1 c(t) &= F^1(k),\ \text{for } t < t_1 \\
\dot{k}(t) + q_2 c(t) &= F^2(k), \text{for } t > t_1, \\
k(0) &= k_0,\ k(t) \geq 0,\ c(t) \geq 0
\end{aligned}$$

is at the basis of the technology adoption problem, where $k(t)$, $c(t)$, and t_1 are choice variables. In this study a more general version of this problem will be considered but this simple setup is useful to understand the basics of adoption problem. This is a representative agent model with intertemporal utility function $\int_0^\infty U(c(t))e^{-\rho t}dt$. The problem is composed of two periods, where each one corresponds to a different mode of technology, t_1 refers to the instant of the switch between these modes, k and c denote capital and consumption, respectively, $F^1(k)$ and $F^2(k)$ are production functions in the respective periods, and $U(c(t))$ is the instantaneous utility function. The obvious problem here is to characterize optimal paths of capital and consumption, together with the switching instant.

The problem rests on the following relationships to hold in each period:

$$\begin{aligned}
y(t) &= F(k(t)) = c(t) + i(t) \\
\dot{k}(t) &= q\ i(t),
\end{aligned}$$

where $i(t)$ denotes investment (we omit depreciation both here and in the application at the third chapter). First equation is the usual resource constraint while the second one denotes the evolution equation of capital. The variable q (q_1, q_2 in respective periods) represents the level of utilized technology. In general available technology may be higher than the utilized one. Indeed, in this setup, it is assumed that at $t = 0$ both technologies, i.e. q_1, q_2, are present, but only after switching q_2 is utilized. Obviously, the central planner has the option not to switch to the higher technology, or switch immediately to the higher technology. These are corner solutions to the problem. In a more general case in which one is allowed to switch more than once, say three, possibility of corner solutions imply that, number of switches is in fact the maximum number of switches allowed. The q here represents embodied technology, that only affects the new capital goods, which hence must be labeled as "investment specific."

But switching to a new technology has its costs which can be very high (see Jovanovic, 1997). These are first, costs due to the obsolescence of existing capital. A reassignment of resources towards capital goods in case of an increase in embodied technology (this will be the case as higher embodied technology decreases relative price of new capital) will induce a drop in consumption, thereby resulting with a loss in welfare. This is referred to as obsolescence cost (see Boucekkine et al., 2003). Loss in specific human capital is also posed as a particular cost in Parente (1994), Greenwood and Jovanovic (2001). This can be interpreted basically in terms of the expertise loss due to the adoption of a new technology. This will be reflected in the production functions of the two periods. For example, having $F^1(k) = a_1 k$ and $F^2(k) = a_2 k$, where $a_2 < a_1$ would reflect these costs. A learning structure in line with Parente (1994) and Boucekkine et.al., (2004) may also imposed in the second period so that expertise loss is overcome in time. An example would be the following production function from Boucekkine et.al., (2004): $F^2(k(t)) = (1 - Ae^{-\theta(t-t_1)})k(t)$.

Given these costs, and the advantage of efficiency in the capital sector by higher level of technology, the trade-off at the basis of technology adoption should be clear by now.

There are few papers dealing with the necessary conditions of optimality for this problem. These papers are Tomiyama (1985), Tomiyama and Rossana (1989), Makris (2001), Sağlam (2002). These papers utilize a dynamic programming approach (principle of optimality) together with standard optimal control techniques. The main idea is to reduce the two stage problem to a standard one, first by solving the second period problem and then attaching it to the first period problem, in order to utilize dynamic programming technique, while Pontryagin Maximum Principle concludes at the intermediate steps. Illustrations of this technique may be found in Sağlam (2002), Boucekkine, et.all (2003), and Karaşahin (2006).

Three important aspects of the problem are first, the horizon of the functional to be maximized (infinite horizon case), second, the dependence of the constraint functions and $U(c(t))$ on the switching instant (we will refer to the situation in this second case as "explicit dependence to the switching instant"), third possibility of more than one switch. There is no paper that deals with all of these at the same time. Makris (2001) considers an infinite horizon problem with multiple switches, yet ignores explicit dependence to the switching instant. Tomiyama and Rossana (1989) develop, Tomiyama (1985) at this last point while working in finite horizon with a single switch. Sağlam (2002) considers multiple switches in finite horizon. So the infinite horizon multi-stage problem with explicit dependence to the switching instant remains to be dealt with.

We should also note that multi-stage problems are not restricted to economics, although we restrict ourselves to economics here. In fact, the recently uprising hybrid optimal control literature focuses on these types of problems occuring in engineering (for a short overview of this literature, see Xu and Antsaklis, 2002). Leaving aside the applied approach in this literature which focuses on developing algoritms for solutions of such problems, the theoretical results in this literature are limited to the problems in which the cost functions and the constraints are invariant under time translations (see Sussmann (1999) and Garavello and Piccoli (2005)). Thus, as it will be more clear when we define the general problem that we will consider in the next chapter, the main type of problem in this study, i.e. problems with "explicit dependence to the switching instant" falls outside the scope of the current theoretical state of hybrid optimal control. Yet the practical results of this literature should be noted as they are related to somehow more general systems then we consider here. In particular, algoritms developed within this tradition applies to the cases when there is no predetermined sequence of subsystems.

Returning to our formulation of the problem, aside from the standard optimality conditions, like Euler-Lagrange equation, which will be shown to hold in our case, two specific necessary conditions occur here. These are nothing but extensions of the Weierstrass-Erdmann corner conditions. We will be able to show that indeed Weierstrass-Erdmann corner conditions extend to the problems with switches.

We proceed in entirely different lines with the existing literature. We treat the problem as an ordinary problem in calculus of variations, and attack it by the standard tools from the calculus of variations. We also utilize some basic properties of Sobolev spaces. Since in optimal growth models the path of capital will be in a Sobolev space (see the third chapter) this is the natural setting for the problem.

In this framework we will be able to extend the necessary conditions of optimality to the cases with multiple switches in infinite horizon under explicit dependence to the switching instant, as this case has never been dealt with before. Moreover we will translate our results into the Hamiltonian "language".

Organization of the paper is as follows: we start with the formulation of a set of necessary conditions of optimality in a two-stage problem, and extend them to the multiple-switch case. Then we discuss the passage from our formulation of the necessary conditions to that of Tomiyama and Rossana (1989) and Makris (2001). Finally, we apply the results obtained to a problem with expanding technology frontier, and conclude.

CHAPTER 2

MAIN RESULTS

2.1 Euler-Lagrange Equation, Continuity Condition

We generalize and rewrite the problem mentioned at the introduction as follows:

$$\max_{x(t),t_1} \int_{t_0}^{t_1} L^1(x,\dot{x},t,t_1)e^{-\rho t}dt + \int_{t_1}^{t_f} L^2(x,\dot{x},t,t_1)e^{-\rho t}dt$$

subject to

$$(x(t),\dot{x}(t)) \in D_{t_1}(t) \subset \mathbb{R}^2,\ x(t_0)=x_0,\ x(t)\geq 0,\ \text{a.e on } [t_0,t_f],\ t_f \leq \infty,$$

where

$$D_{t_1}(t) = \{(x,y) \mid f^1(x,y,t,t_1) \geq 0,\ \text{for } t_0 \leq t < t_1;$$
$$f^2(x,y,t,t_1) \geq 0, \text{for } t_f \geq t > t_1 \ \}$$

We also assume that t_0 and t_f are fixed. Although we write the integrands and the constraint set as if they depend on the switching instant, t_1, this need not be the case. But whenever this is the case, we will generally say that "the problem depends explicitly to the switching instant", as we have pointed out at the introduction. Throughout this study, t_1 will refer to the optimal switching instant, and whenever we say that x is optimal, we will mean the x in the solution pair (x,t_1).

First we state some preliminary material that we will utilize throughout this study, from Brezis (1983). We will say that a function, $x : [t_0, t_f] \to \mathbb{R}$, is *locally integrable*, and write $x \in L_1(loc)$, if it is integrable on any bounded interval (the space L_1 will be the space of integrable functions). $L^\infty(loc)$ will denote functions essentially bounded on finite intervals. By $x \in C_c^k(a, b)$, for (a, b) an open interval, we will mean that $x \in C^k(a, b)$, i.e. x is k-th times continuously differentiable, and $\operatorname{supp} x = \overline{\{t \in \mathbb{R}_+ : x(t) > 0\}} \subset (a, b)$. For any $x \in L_1(loc)$, we will say and write that x' is the *weak derivative* of x, if $\forall h \in C_c^1(t_0, t_f)$, $\int_{t_0}^{t_f} x(t)\dot{h}(t) = -\int_{t_0}^{t_f} x'(t)h(t)$. For a function $x \in C_c^1(t_0, t_f)$, the weak derivative will be identical with the ordinary derivative.

The space $W^{1,1} \equiv W^{1,1}(t_0, t_f) \equiv \{x \in L_1 : x' \text{ exists and } x' \in L_1\}$ will be the *Sobolev space* that we will frequently be referring to. $W^{1,1}(loc)$ is similarly defined on (t_0, t_f) to be $\{x \in L_1(loc) : x' \text{ exists and } x' \in L_1(loc)\}$. Two important properties of this space will prove to be useful here. First one is that for any function x in $W^{1,1}$, as the elements of this space are equivalence classes, there is a continuous representative, say $\bar{x}$, which is equal to x almost everywhere. So we will be talking about this representative, whenever we refer to an element of this space. Second, relatedly, weak derivative coincides with the usual derivative almost everywhere and $\bar{x}(b) = \bar{x}(a) + \int_a^b x' dt$. Thus elements of this space are absolutely continuous functions on finite intervals. In fact, on a finite open interval, the set of absolutely continuous functions and the space $W^{1,1}$ are the same.

We will have the following assumptions. From now on, third and fourth arguments of L will be suppressed, whenever we do not need them. Moreover, x will refer to the optimal x, always, unless otherwise stated.

A1 $L^i(.) : \mathbb{R}^4 \to \mathbb{R}$ is C^1; $f^i(.) : \mathbb{R}^4 \to \mathbb{R}$ is continuous, for $i = 1, 2$.

A2 EXISTENCE There is an optimal pair $(x(t), t_1)$ that solves the above problem with $\int_{t_0}^{t_1} L^1(x, \dot{x})e^{-\rho t}dt + \int_{t_1}^{t_f} L^2(x, \dot{x})e^{-\rho t}dt < \infty$.

A3 INTERIORITY $x(t) > 0$, $f^i(x, y, t, t_1) > 0$ uniformly (in the sense of the space L^∞) on any bounded interval, for $i = 1, 2$.

A4 $x(t) \in W^{1,1}(loc)$; $\dot{x}(t) \in L^\infty(loc)$.

Our first result will be a Euler-Lagrange equation that fits our purposes here. This equation is rather standard. The only nonstandard thing here is the change in the objective functional at an instant.

Proposition 1 *EULER LAGRANGE EQUATION Optimal $x(t)$ satisfies:*

$$(L_{\dot{x}}(x, \dot{x})e^{-\rho t})' = L_x(x, \dot{x})e^{-\rho t} \tag{2.1}$$

on any bounded interval (a, b), under the assumptions A1,A3,A4 (L should be read as L^1 whenever $t < t_1$, as L^2 whenever $t > t_1$; ' denotes the weak derivative, as we have noted above).

Proof Consider any bounded interval (a, b) on (t_0, t_f).Take any $h \in C_c^1(a, b)$, and assume that it is extended to zero outside (a, b).

For λ small $x + \lambda h > 0$, clearly. Moreover, for λ small, for an appropriate ϵ, $(x + \lambda h, \dot{x} + \lambda\dot{h})$ is in an open ball of radius ϵ centered at $(x, \dot{x})$, for each $t \in (a, b)$ so that $f^i(x + \lambda h, \dot{x} + \lambda\dot{h}, t, t_1) > 0$, for $i = 1, 2$.

Define $\varphi(\lambda) = \int_a^b L(x + \lambda h, \dot{x} + \lambda\dot{h})e^{-\rho t}dt = \int_a^{t_1} L^1(x + \lambda h, \dot{x} + \lambda\dot{h})e^{-\rho t}dt + \int_{t_1}^b L^2(x + \lambda h, \dot{x} + \lambda\dot{h})e^{-\rho t}dt$, and write $\varphi_1(\lambda) = \int_a^{t_1} L^1(x + \lambda h, \dot{x} + \lambda\dot{h})e^{-\rho t}dt$, $\varphi_2(\lambda) = \int_{t_1}^b L^2(x + \lambda h, \dot{x} + \lambda\dot{h})e^{-\rho t}dt$. For any sequence of real numbers $\lambda_n \to 0$, fixing any t,

$$\frac{L(x + \lambda_n h, \dot{x} + \lambda_n\dot{h}) - L(x, \dot{x})}{\lambda_n} = L_x(x + \bar{\lambda}_n h, \dot{x} + \bar{\lambda}_n\dot{h})h + L_{\dot{x}}(x + \bar{\lambda}_n h, \dot{x} + \bar{\lambda}_n\dot{h})\dot{h},$$

for some $0 < \bar{\lambda}_n < \lambda_n$, by mean value theorem. Now, L_x and $L_{\dot{x}}$ are continuous, and are restricted here to a bounded rectangle in $\mathbb{R}^2$, due the the continuity of x and boundedness of $\dot{x}$. So, $L_x(x + \bar{\lambda}_n h, \dot{x} + \bar{\lambda}_n\dot{h})$, and

$L_{\dot{x}}(x+\bar{\lambda}_n h, \dot{x}+\bar{\lambda}_n \dot{h})\dot{h}$ are bounded in $L^\infty(a,b)$. Thus, there is $K \in \mathbb{R}$ such that $\left|\frac{L(x+\lambda_n h, \dot{x}+\dot{\lambda_n h}) - L(x,\dot{x})}{\lambda_n}\right| \leq K$, *a.e.* But then, we may apply Dominated Convergence Theorem to the sequence

$\dfrac{\varphi_1(\lambda_n)-\varphi_1(0)}{\lambda_n} = \displaystyle\int_a^{t_1} \frac{L^1(x+\lambda_n h, \dot{x}+\dot{\lambda_n h}) - L^1(x,\dot{x})}{\lambda_n} e^{-\rho t}dt$, concluding that $\varphi_1(\lambda)$ is differentiable at 0 with the derivative,

$$\lim_{n\to\infty}\int_a^{t_1} \frac{L^1(x+\lambda_n h, \dot{x}+\dot{\lambda_n h}) - L^1(x,\dot{x})}{\lambda_n} e^{-\rho t}dt = \int_a^{t_1}\left(L_x^1(x,\dot{x})he^{-\rho t} + L_{\dot{x}}^1(x,\dot{x})\dot{h}e^{-\rho t}\right)dt.$$

By repeating the same steps on (t_1,b) one may also find that $\varphi_2(\lambda) = \int_{t_1}^b \left(L_x^2(x,\dot{x})he^{-\rho t} + L_{\dot{x}}^2(x,\dot{x})\dot{h}e^{-\rho t}\right)dt$. Hence we easily have that $\varphi'(0) = \int_a^b L_x(x,\dot{x})he^{-\rho t} + L_{\dot{x}}(x,\dot{x})\dot{h}e^{-\rho t}dt$.

Now, $\int_0^\infty L(x+\lambda h, \dot{x}+\lambda\dot{h})e^{-\rho t}dt - \int_0^\infty L(x,\dot{x})e^{-\rho t}dt = \varphi(\lambda)-\varphi(0)$, so that $\varphi(.)$ is maximized at 0. Since $\varphi(.)$ is differentiable at zero,

$$\varphi'(0) = \int_a^b (L_x(x,\dot{x})e^{-\rho t}h + L_{\dot{x}}(x,\dot{x})e^{-\rho t}\dot{h})dt = 0. \tag{2.2}$$

As $h \in C_c^1(a,b)$ was arbitrary, $(L_{\dot{x}}(x,\dot{x})e^{-\rho t})' = L_x(x,\dot{x})e^{-\rho t}$, i.e. $L_x(x,\dot{x})e^{-\rho t}$ is the weak derivative of $L_{\dot{x}}(x,\dot{x})e^{-\rho t}$ on (a,b). □

As a result of this proposition, we will have the first important necessary condition for problems with switches. This is also called the first Weierstrass-Erdmann corner condition.

Corollary 1 *CONTINUITY CONDITION Assumption 4 with EL equation imply that* $L_{\dot{x}}(x,\dot{x})e^{-\rho t} \in W^{1,1}(loc)$,*hence is absolutely continuous on any bounded interval, hence continuous everywhere. In particular we have continuity at the switching instant.*

2.2 Characterization of The Switching Instant

Before establishing the optimality condition with respect to the switching instant, we need to impose more regularity on the solution $x(t)$. We do this as follows:

Corollary 2 *Optimal x is locally Lipschitz, i.e. it is Lipschitz on any bounded interval.*

Proof This is immediate since $x \in W^{1,1}(loc)$, and $|\dot{x}|$ is bounded. Indeed, for any $a, b \in (t_0, t_f)$, if for some K, $|\dot{x}| \leq K$, then $x(b) - x(a) = \int_a^b \dot{x} dt \leq K |b - a|$. $\square$

If we continue to rely on the assumptions stated above, then we have the following. We need below, in particular, the Euler-Lagrange equation.

Proposition 2 *If the optimal x is Lipschitz on an open interval I, $L_{\dot{x}}$ is C^1 (thus switching point should not be in this interval, in general), and $L_{\dot{x}\dot{x}} < 0$ on its entire domain, then x is C^2 on $\bar{I}$. Thus x is C^2 except possibly at t_1.*

Proof See Butazzo, et al. (1998), Proposition 4.4, page 135. $\square$

A5 ADMISSIBILITY $\exists \delta > 0, T \in \mathbb{N}$ such that $\forall\, \varepsilon \in (-\delta, \delta)$, $\forall t \geq T$, $f^i(x, \dot{x}, t, t_1 + \varepsilon) \geq 0$, for $i \in \{1, 2\}$, where (x, t_1) denotes an optimal pair, as usual.

A6 There exist an integrable function $g(t)$ on $[t_0, t_f]$ and some $\delta > 0$: $\forall\, \varepsilon \in (-\delta, \delta)$, $\left| L^i_{t_1}(x, \dot{x}, t, t_1 + \varepsilon) \right| \leq g(t)$ for $i \in \{1, 2\}$.

A7 $\ddot{x} \in L^1(loc)$.

Below proposition, which is a variant of the second Weierstrass-Erdmann corner conditions, will be proved by the so-called "variation of the independent variable" technique. The assumptions above are crucial for this result.

By A5 we will be able to work in infinite horizon. It is stated at the most general level required by the proof below. This assumption, together with A6 enable us to derive the derivative at (2.4), and equate it to zero. Finally, A7 is required to carry out the integration by parts step below, as it ensures absolute continuity of the functions involved, together with A4.

In order to be neat, we will sometimes write limit of an expression with a subscript attached to that expression, showing the point, and direction of the limit.

Proposition 3 *MATCHING CONDITION Under the assumptions above the optimal pair* $(x(t), t_1)$ *satisfies*

$$[\dot{x}L^1_{\dot{x}} - L^1]_{t_1-} e^{-\rho t_1} - [\dot{x}L^2_{\dot{x}} - L^2]_{t_1+} e^{-\rho t_1} = \int_{t_0}^{t_1} L^1_{t_1} e^{-\rho t} dt + \int_{t_1}^{t_f} L^2_{t_1} e^{-\rho t} dt \quad (2.3)$$

whenever $t_0 < t_1 < t_f$.

Proof Take any $h \in C^1_c(t_0, t_f)$, and define the mapping on $[t_0, t_f]$ by $\tau(t, \epsilon) = t - \epsilon h(t) \equiv s$ (h is extended to zero outside (t_0, t_f)). For ϵ small enough, $\tau_t(t, \epsilon) = 1 - \epsilon h'(t) > 0$ (we continue to use subscripts for derivatives). Thus for any ϵ, the mapping $\tau(., \epsilon)$ is a C^1 diffeomorphism of $[t_0, t_f]$. Write $\zeta(s, \epsilon)$, for the inverse of this mapping, and $\tau(t_1, \epsilon) = s_1$. Since the transformation $t \mapsto s$, is monotonic, for ϵ small enough, the path $x(\zeta(s, \epsilon))$ as a function of s, satisfies the constraints of the problem, where s_1 is the instant of switch (if $t_f = \infty$, we need A5 to ensure admissibility). So, $\varphi(\epsilon) = \int_{t_0}^{s_1} L^1(x(\zeta(s, \epsilon)), \frac{dx(\zeta(s,\epsilon))}{ds}, s, s_1)dt + \int_{s_1}^{t_f} L^2(x(\zeta(s, \epsilon)), \frac{dx(\zeta(s,\epsilon))}{ds}, s, s_1)ds$ is maximized at 0 (we assume that the term $e^{-\rho t}$ is subsumed under the functions L^i, for ease of demonstration, also note that $\tau(t, 0) = t$). Since $\dfrac{dx(\zeta(s, \epsilon))}{ds} = \dot{x}(\zeta(s, \epsilon))\zeta_s(s, \epsilon)$, we write:

$$\begin{aligned} \varphi(\epsilon) &= \int_{t_0}^{s_1} L^1(x(\zeta(s, \epsilon)), \dot{x}(\zeta(s, \epsilon))\zeta_s(s, \epsilon), s, s_1)ds \\ &\qquad + \int_{s_1}^{t_f} L^2(x(\zeta(s, \epsilon)), \dot{x}(\zeta(s, \epsilon))\zeta_s(s, \epsilon), s, s_1)ds. \end{aligned} \quad (2.4)$$

$\varphi(\epsilon)$ is finite and τ is a C^1 diffeomorphism, so by an application of change of variables (Lang, 1993, p.505, Theorem 2.6) we transform the above function as:

$$\begin{aligned}\varphi(\epsilon) \;&= \textstyle\int_{t_0}^{t_1} L^1(x(t), \dot{x}(t)\frac{1}{\tau_t(t,\epsilon)}, \tau(t,\epsilon), \tau(t_1,\epsilon))\tau_t(t,\epsilon)dt \\ &\quad + \textstyle\int_{t_1}^{t_f} L^2(x(t), \dot{x}(t)\frac{1}{\tau_t(t,\epsilon)}, \tau(t,\epsilon), \tau(t_1,\epsilon))\tau_t(t,\epsilon)dt\end{aligned} \tag{2.5}$$

where we use $\tau_t(\zeta(s,\epsilon),\epsilon)\zeta_s(s,\epsilon) = 1$.

Now, in a neighborhood of zero, by assumptions A1, A4, and A6, the partial derivatives with respect to ϵ of the integrands above, $(1-\epsilon h')[-L^i_t h + \dot{x}L^i_{\dot{x}}\frac{h'}{(1-\epsilon h')^2} - L^i_{t_1}h(t_1)] - L^i h'$, will be dominated by an integrable function, from which it will follow by dominated convergence theorem that $\varphi(\epsilon)$ is differentiable at zero (we suppress arguments of the functions). This derivative is zero, and given by the following expression :

$$\begin{aligned}\varphi'(0) = \;& \int_{t_0}^{t_1} [-L^1_t h + \dot{x}L^1_{\dot{x}}h' - L^1_{t_1}h(t_1) - L^1 h']\, dt \\ &\quad + \int_{t_1}^{t_f} [-L^2_t h + \dot{x}L^2_{\dot{x}}h' - L^2_{t_1}h(t_1) - L^2 h']\, dt.\end{aligned} \tag{2.6}$$

By integration by parts $\int_{t_0}^{t_1} [\dot{x}L^1_{\dot{x}} - L^1]h'dt = [\dot{x}L^1_{\dot{x}} - L^1]_{t_1-}h(t_1) - \int_{t_0}^{t_1} \frac{d[\dot{x}L^1_{\dot{x}}-L^1]}{dt}hdt$, and $\int_{t_1}^{t_f} [\dot{x}L^2_{\dot{x}} - L^2]h'dt = \int_{t_1}^{b} [\dot{x}L^2_{\dot{x}} - L^2]h'dt = -[\dot{x}L^2_{\dot{x}} - L^2]_{t_1+}h(t_1) - \int_{t_1}^{b} \frac{d[\dot{x}L^2_{\dot{x}}-L^2]}{dt}hdt$, for some $b < \infty$, as h has a compact support (see that $\dot{x}L^1_{\dot{x}} - L^1$ and $\dot{x}L^2_{\dot{x}} - L^2$ are absolutely continuous on $[0, t_1]$, $[t_1, b]$, respectively, so that integration by parts is possible, by Gordon (1994), p. 185, Theorem 12.5). Plugging these in $\varphi'(0)$ we have $h(t_1)\left([\dot{x}L^1_{\dot{x}} - L^1]_{t_1-} - [\dot{x}L^2_{\dot{x}} - L^2]_{t_1+}\right) + \int_{t_0}^{t_1}(-L^1_t - \frac{d[\dot{x}L^1_{\dot{x}}-L^1]}{dt})hdt + \int_{t_1}^{t_f}(-L^2_t - \frac{d[\dot{x}L^2_{\dot{x}}-L^2]}{dt})hdt = h(t_1)\left(\int_{t_0}^{t_1} L^1_{t_1}dt + \int_{t_1}^{t_f} L^2_{t_1}dt\right)$. For $h(t_1) \neq 0$,

$$[\dot{x}L^1_{\dot{x}}-L^1]_{t_1-}-[\dot{x}L^2_{\dot{x}}-L^2]_{t_1+}=\left(\int_{t_0}^{t_1}L^1_{t_1}dt+\int_{t_1}^{t_f}L^2_{t_1}dt\right)$$

$$+\frac{1}{h(t_1)}\left[\int_{t_0}^{t_1}(L^1_t+\tfrac{d[\dot{x}L^1_{\dot{x}}-L^1]}{dt})hdt+\int_{t_1}^{t_f}(L^2_t+\tfrac{d[\dot{x}L^2_{\dot{x}}-L^2]}{dt})hdt\right] \tag{2.7}$$

It follows that the expression on the far most right is constant for any λ with $\lambda(t_1)\neq 0$. But it is also linear in λ, so it must be zero. $\square$

For the corner solutions we need to modify the above condition as:

$$[\dot{x}L^1_{\dot{x}}-L^1]_{t=t_0}e^{-\rho t_0}-[\dot{x}L^2_{\dot{x}}-L^2]_{t=t_0}e^{-\rho t_0}\geq\int_{t_0}^{t_f}L^2_{t_1}e^{-\rho t}dt$$

and

$$[\dot{x}L^1_{\dot{x}}-L^1]_{t\to t_f}e^{-\rho t_f}-[\dot{x}L^2_{\dot{x}}-L^2]_{t\to t_f}e^{-\rho t_f}\leq\int_{t_0}^{t_f}L^1_{t_1}e^{-\rho t}dt$$

for $t_1=0$, and $t_1\to\infty$, respectively. These follow from the requirement that $\lim_n\varphi'(0,t^n)\leqslant 0$, as $t^n\downarrow t_1=t_0$, and $t^n\uparrow t_1=\infty$, respectively. For the proof of the case $t_1=t_0$, we define $\varphi(\epsilon,t^n)$ as in (2.4), where s_1 is replaced by $s_n=\tau(t^n,\epsilon)$, and by $\varphi'(\epsilon,t^n)$, we mean derivative with respect to the first variable:

$$\begin{aligned}\varphi(\epsilon,t^n)=\;&\int_{t_0}^{s_n}L^1(x(\zeta(s,\epsilon)),\dot{x}(\zeta(s,\epsilon))\zeta_s(s,\epsilon),s,s_n)ds\\&+\int_{s_n}^{t_f}L^2(x(\zeta(s,\epsilon)),\dot{x}(\zeta(s,\epsilon))\zeta_s(s,\epsilon),s,s_n)ds.\end{aligned}$$

As we need the result of a limit in which $t^n\downarrow t_1=t_0$, the setting in the proof above dictates that we consider $\epsilon\downarrow 0$, where $h_n(t^n)<0$, so that $\tau(t^n,\epsilon)=t^n-\epsilon h_n(t^n)\downarrow t_0$, as $\epsilon\downarrow 0$, $n\to\infty$ (sure one may assume $\epsilon\uparrow 0$, while $h_n(t^n)>0$). We write h_n instead of h as we will also impose $h_n(t^n)=-1$, for each $n\in\mathbb{N}$, and $\operatorname{supp}h_n\downarrow 0$, as $n\to\infty$ (since we consider the case $t_n\downarrow t_0$, $\operatorname{supp}h_n\downarrow 0$ is automatically true). Now, for the validity of the argument $\lim_n\varphi'(0,t^n)\leqslant 0$, first check that we are able to differentiate $\varphi(\epsilon)$ at zero even if we replace t_1 in the definition of $\varphi(\epsilon)$ with t^n close to t_0, and by the continuity of

$$\begin{aligned}\varphi'(0,t^n) &= \int_{t_0}^{t^n} [-L_t^1 h + \dot{x}L_{\dot{x}}^1 h' - L_{t_1}^1 h(t_n) - L^1 h']\,dt \\ &\quad + \int_{t^n}^{t_f} [-L_t^2 h + \dot{x}L_{\dot{x}}^2 h' - L_{t_1}^2 h(t_n) - L^2 h']\,dt\end{aligned}$$

in t^n (as integrands are bounded around t_0), $\lim \varphi'(0,t^n)$ exists. For the inequality first note

$$\begin{aligned}\varphi(\epsilon,t^n) - \varphi(0,t^n) = &\int_{t_0}^{t^n} L^1(x(t),\dot{x}(t)\frac{1}{\tau_t(t,\epsilon)},\tau(t,\epsilon),\tau(t_n,\epsilon))\tau_t(t,\epsilon)dt \\ &+ \int_{t^n}^{t_f} L^2(x(t),\dot{x}(t)\frac{1}{\tau_t(t,\epsilon)},\tau(t,\epsilon),\tau(t_1,\epsilon))\tau_t(t,\epsilon)dt \\ &- \int_{t_0}^{t^n} L^1(x(t),\dot{x}(t),t,t_n)dt - \int_{t^n}^{t_f} L^2(x(t),\dot{x}(t),t,t_n)dt \quad (2.8)\end{aligned}$$

So,

$$\begin{aligned}&\frac{\varphi(\epsilon,t^n)-\varphi(0,t^n)}{\epsilon} \\ &= \int_{t_0}^{t^n} \frac{L^1(x(t),\dot{x}(t)\frac{1}{\tau_t(t,\epsilon)},\tau(t,\epsilon),\tau(t_1,\epsilon))\tau_t(t,\epsilon) - L^1(x(t),\dot{x}(t),t,t_n)}{\epsilon}dt \\ &+ \int_{t^n}^{t_f} \frac{L^2(x(t),\dot{x}(t)\frac{1}{\tau_t(t,\epsilon)},\tau(t,\epsilon),\tau(t_1,\epsilon))\tau_t(t,\epsilon) - L^2(x(t),\dot{x}(t),t,t_n)}{\epsilon}dt\end{aligned}$$

By mean value theorem, for some $0 < \delta(t) < \epsilon$, and

$$c_n(t) \equiv (x(t),\dot{x}(t)\frac{1}{\tau_t(t,\delta(t))},\tau(t,\delta(t)),\tau(t_n,\delta(t)))\tau_t(t,\epsilon),$$

$$\begin{aligned}&\frac{\varphi(\epsilon,t^n)-\varphi(0,t^n)}{\epsilon} \\ &= \int_{t_0}^{t^n} \Big[\Big\{- L_t^1(c_n(t))h + \dot{x}L_{\dot{x}}^1(c_n(t))\tfrac{h'}{(1-\epsilon h')^2} - L_{t_1}^1(c_n(t))h(t_n)\Big\}\tau_t(t,\epsilon) - L^1(c_n(t))h'\Big]dt \\ &+ \int_{t^n}^{t_f} \Big[\Big\{- L_t^2(c_n(t))h + \dot{x}L_{\dot{x}}^2(c_n(t))\tfrac{h'}{(1-\epsilon h')^2} - L_{t_1}^2(c_n(t))h(t_n)\Big\}\tau_t(t,\epsilon) - L^2(c_n(t))h'\Big]dt\end{aligned}$$

Plugging $h(t_n) = -1$, and $\tau_t(t, \epsilon) = 1 - \epsilon h'(t)$, and omitting the terms including $\epsilon h'(t)$ in the product of the curly brackets above (these will disappear in the limit considered below, so for ease of demonstration we will exclude them), we have:

$$
\begin{aligned}
&\left| \frac{\varphi(\epsilon, t^n) - \varphi(0, t^n)}{\epsilon} - \varphi'(0, t^n) \right| \\
&\quad \leq \int_{t_0}^{t^n} \left| \left(L_t^1 - L_t^1(c_n(t)) \right) \ h \right| \, dt + \int_{t_0}^{t^n} \left| \left(\dot{x} L_{\dot{x}}^1(c_n(t)) \frac{1}{(1-\epsilon h')^2} - \dot{x} L_{\dot{x}}^1 \right) \ h' \right| dt \\
&\quad - \int_{t_0}^{t^n} \left| L_{t_1}^1 - L_{t_1}^1(c_n(t)) \right| dt + \int_{t_0}^{t^n} \left| \left(L^1 - L^1(c_n(t)) \right) \ h' \right| dt \\
&\quad + \int_{t^n}^{t_f} \left| \left(L_t^2 - L_t^2(c_n(t)) \right) \ h \right| \, dt + \int_{t^n}^{t_f} \left| \left(\dot{x} L_{\dot{x}}^2(c_n(t)) \frac{1}{(1-\epsilon h')^2} - \dot{x} L_{\dot{x}}^2 \right) \ h' \right| dt \\
&\quad - \int_{t^n}^{t_f} \left| L_{t_1}^2 - L_{t_1}^2(c_n(t)) \right| dt + \int_{t^n}^{t_f} \left| \left(L^2 - L^2(c_n(t)) \right) \ h' \right| dt
\end{aligned}
$$

The integrands including h or h', can be made arbitrarily small for any $\epsilon > 0$, by choosing n large, since compact support of h or h' allows us to work on compact sets, and since the integrands are continuous. So the problem is with the terms $\int_{t_0}^{t^n} \left| L_{t_1}^1 - L_{t_1}^1(c_n(t)) \right| dt$, $\int_{t^n}^{t_f} \left| L_{t_1}^2 - L_{t_1}^2(c_n(t)) \right| dt$. The first one of these terms is easy to handle since the integrand is continuous and restricted to compact set whose measure decreases to zero, as $n \to \infty$. Hence taking n large enough is sufficient, to make it arbitrarily small. For the second one, by the integrability assumption, A6, we may restrict our attention to a compact subset of $[t_0, t_f]$. On this subset, the set $\operatorname{supp} h_n$ can also be handled easily. So, it remains to show that the term, for some $b < \infty$,

$$
\int_{t_0}^{b} \left| L_{t_1}^2(x(t), \dot{x}(t), t, t_n) - L_{t_1}^2(x(t), \dot{x}(t), t, t_n + \epsilon) \right| dt
$$

is arbitrarily small for all n large (note that outside the support of h, $c_n(t) = (x(t), \dot{x}(t), t, t_n + \epsilon))$. But this is obvious since, $L_{t_1}^2(x(t), \dot{x}(t), t, t_n)$ is uniformly continous on $[t_0, b]$. Restricting ϵ to some sufficiently small values, the integrand, hence the integral will be small for all n. Thus we may state the following: for any $\alpha > 0$, $\exists N : \forall n \geq N$, $\left|\frac{\varphi(\epsilon,t^n)-\varphi(0,t^n)}{\epsilon} - \varphi'(0,t^n)\right| < \frac{\alpha}{2}$, for all ϵ small.

Now, if $\lim_n \varphi'(0,t^n) > 0$ is the case, one may take α also satisfying $\lim_n \varphi'(0,t^n) \geq \alpha > 0$. So $\frac{\varphi(\epsilon,t^n)-\varphi(0,t^n)}{\epsilon} \geq \frac{\alpha}{2}$, for all ϵ small, $n \geq N$. Taking limit with respect to n gives $\varphi(\epsilon,t_0) - \varphi(0,t_0) > 0$, which contradicts the optimality of $t_1 = t_0$.Hence one obtains (2.7) with equality replaced with $\geq$. In this case also, linearity of the right most expression in h implies that it is a constant. Hence comes the result.

For the case $t_1 = t_f$, the same arguments apply with little change.

2.3 Extension to the Multiple Switch Case

We will show in this section that above results are easily generalized to the multiple switch case. The problem we have in mind is below. It is sufficient to consider a problem with two switches. The assumptions for single switch case generalize to this case, easily, so we will not deal with them.

$$\max_{x(t),t_1} \int_{t_0}^{t_1} L^1(x,\dot{x},t,t_1,t_2)e^{-\rho t}dt + \int_{t_1}^{t_2} L^2(x,\dot{x},t,t_1,t_2)e^{-\rho t}dt + \int_{t_2}^{t_f} L^3(x,\dot{x},t,t_1,t_2)e^{-\rho t}dt$$

subject to

$$(x(t),\dot{x}(t)) \in D_{t_1,t_2}(t) \subset \mathbb{R}^2, x(0) = x_0,\ x(t) \geq 0,\ \text{a.e on } [0,\infty),\ t_f \leq \infty,$$

$$\begin{aligned} D_{t_1,t_2}(t) &= \{(x,y) \mid f^1(x,y,t,t_1,t_2) \geq 0,\ \text{for } t < t_1;\ f^2(x,y,t,t_1,t_2) \geq 0, \\ &\qquad \text{for } t_2 > t > t_1;\ f^3(x,y,t,t_1,t_2) \geq 0, \text{ for } t_f > t > t_2\}. \end{aligned}$$

Assume that (x,t_1,t_2) is a solution to the problem. Euler-Lagrange equation, and hence continuity condition extends immediately to this case. So we will deal with the extension of the matching condition. By the proof of the matching condition, it is clear that we can rewrite (2.7) as:

$$\varphi'(0) = h(t_f)[\dot{x}L^3_{\dot{x}} - L^3]_{t_f-} - h(t_2)[\dot{x}L^3_{\dot{x}} - L^3]_{t_2+} - \int_{t_2}^{t_f} \big(h(t_1)L^3_{t_1} + h(t_2)L^3_{t_2} + \phi^3 h\big)\, dt$$
$$+h(t_2)[\dot{x}L^2_{\dot{x}} - L^2]_{t_2-} - h(t_1)[\dot{x}L^2_{\dot{x}} - L^2]_{t_1+} - \int_{t_1}^{t_2} \big(h(t_1)L^2_{t_1} + h(t_2)L^2_{t_2} + \phi^2 h\big)\, dt$$
$$+h(t_1)[\dot{x}L^1_{\dot{x}} - L^1]_{t_1-} - h(t_0)[\dot{x}L^1_{\dot{x}} - L^1]_{t_0+} - \int_{t_0}^{t_1} \big(h(t_1)L^1_{t_1} + h(t_2)L^1_{t_2} + \phi^1 h\big)\, dt$$

where $\phi^i(t) \equiv -L^i_t - \dfrac{d[\dot{x}L^i_{\dot{x}} - L^i]}{dt}$, for $i \in \{1, 2, 3\}$.

For $t_0 < t_1 < t_2 < t_f$, we have $\varphi'(0) = 0$. Now, if h is such that $h(t_1) \neq 0$, $h(t_2) = 0$, $(h(t_f) = h(t_0) = 0$, as h will have compact support on $(t_0, t_f))$, then:

$$[\dot{x}L^1_{\dot{x}} - L^1]_{t_1-} e^{-\rho t_1} - [\dot{x}L^2_{\dot{x}} - L^2]_{t_1+} e^{-\rho t_1} = \int_{t_0}^{t_1} L^1_{t_1} e^{-\rho t} dt + \int_{t_1}^{t_f} L^2_{t_1} e^{-\rho t} dt + \int_{t_2}^{t_f} L^3_{t_1} dt.$$

And similarly,

$$[\dot{x}L^2_{\dot{x}} - L^2]_{t_2-} e^{-\rho t_1} - [\dot{x}L^3_{\dot{x}} - L^3]_{t_2+} e^{-\rho t_1} = \int_{t_0}^{t_1} L^1_{t_2} e^{-\rho t} dt + \int_{t_1}^{t_f} L^2_{t_2} e^{-\rho t} dt + \int_{t_2}^{t_f} L^3_{t_2} dt.$$

These are the necessary conditions for t_1, t_2 be interior optimal switching instants. For the corner conditions, first check that there are seven possible corner solutions. In general, in a system with k switches there will be $2^{k+1} - 1$ possible corner solutions. But it is not hard to adapt the conditions in the previous section here. As an example we will take the following out of the seven configuration: $t_0 = t_1 = t_2 < t_f$. In this case the system immediately jumps to the third stage. By the same arguments in the previous section, considering the appropriate limits, we have the following as necessary conditions:

$$[\dot{x}L^1_{\dot{x}} - L^1]_{t=t_0} e^{-\rho t_0} - [\dot{x}L^3_{\dot{x}} - L^3]_{t=t_0} e^{-\rho t_0} \geq \int_{t_0}^{t_f} L^3_{t_1} e^{-\rho t} dt, \qquad (2.9)$$

$$[\dot{x}L^2_{\dot{x}} - L^2]_{t=t_0} e^{-\rho t_0} - [\dot{x}L^3_{\dot{x}} - L^3]_{t=t_0} e^{-\rho t_0} \geq \int_{t_0}^{t_f} L^3_{t_2} e^{-\rho t} dt. \qquad (2.10)$$

One may also obtain these in a more intuitive way. As the system jumps immediately to the third stage, in a system with one switch, in which the first system is given by L^1, and by the corresponding constraints (and t_2 is defined to be t_0), we would have an immediate jump to the third system. Thus follows (2.9). Similarly, if the first system is defined to be L^2, then we obtain (2.10).

In this manner, the necessary conditions for all corner solutions can be written. But it is clear that implementing these in practice is really hard, as the number of necessary conditions grow very fast. For example in a three switch system, one would have 15 possible corner solutions, with 3 necessary conditions for each of them (one for each switch) with a total of 45 conditions.

2.4 A Comparison with Optimal Control Approach

In this section we will translate our findings in the previous sections into the Hamiltonian "language" as we also want to confirm our findings by relying on the Hamiltonian based results. We also aim here to highlight the relation between our approach and that of the literature. To this end we will take Tomiyama and Rossana (1989) as main reference and refer to Makris (2001) whenever it is necessary. But this section will not carry the generality we have established in the previous sections. We will restrict ourselves to the generality in the literature while translating our results.

Tomiyama and Rossana (1989) follow a two stage dynamic optimization procedure in order to formulate necessary conditions to this problem. We will summarize their approach but first we need some definitions:

For x, u state and control variables, respectively, the problem is to maximize, for $t_f < \infty$,

$J = \int_{t_0}^{t_1} L^1(x,u,t,t_1)e^{-\rho t}dt + \int_{t_1}^{t_f} L^2(x,u,t,t_1)e^{-\rho t}dt$, where $x(t_0) = x_0$,

$$\dot{x}(t) = \begin{cases} f^1(x,u,t,t_1), & t_0 \le t < t_1 \\ \\ f^2(x,u,t,t_1), & t_1 < t \le t_f \end{cases} \tag{2.11}$$

The Hamiltonians for the first and second stages, namely for the first period, $i = 1$, at which $t \in [t_0, t_1)$ and the second period, $i = 2$, at which $t \in (t_1, t_f]$ are:

$$H^i(x,c,p,t,t_1) = -L^i(x,c,t,t_1)e^{-\rho t} + p(t)f^i(x,c,t,t_1).$$

The value functions for the two stages are:

$$\begin{aligned} J_1(x(t_1),t_1) &= \int_{t_0}^{t_1} L^1(x,c,t,t_1)e^{-\rho t}dt, \\ J_2(x(t_1),t_1) &= \int_{t_1}^{t_f} L^2(x,c,t,t_1)e^{-\rho t}dt, \end{aligned}$$

where x, c, t_1 are optimal.

Now, at the first step one considers the second stage problem, for an arbitrary initial value of state, and an arbitrary switching instant, say $x(t_1)$ and t_1, and obtains by Pontryagin's Maximum Principle, a value function, $J_2(x(t_1), t_1)$, depending on $x(t_1)$ and t_1. Given this, the main problem reduces to maximizing $\int_{t_0}^{t_1} L^1(x,u,t,t_1)e^{-\rho t}dt + J_2(x(t_1),t_1)$, where t_1 and $x(t_1)$ are free. Within this setup comes the standard Pontryagin conditions, i.e. minimization of the Hamiltonians together with the equations

$$\frac{\partial H^i}{\partial p} = \dot{x}, \frac{\partial H^i}{\partial x} = -\dot{p}, \tag{2.12}$$

to be satisfied at each period. Moreover conditions specific to the switching instant are obtained. These are the ones that we have proven above, namely continuity and matching conditions. The approach in Makris (2001) is the same, while he works in infinite horizon. We will now compare the results of Tomiyama and Rossana (1989), Makris (2001) with ours, and discuss them.

The correspondent formulation of the continuity of $L_{\dot{x}}$ at the switching instant in these papers is continuity of the co-state variable $p(t)$, at the switching instant. Their formulation rests on the following observation:

$$\frac{\partial(J_1^*(x(t_1),t_1)+J_2^*(x(t_1),t_1))}{\partial x(t_1)}=0. \tag{2.13}$$

Indeed, assuming the differentiability of J_1^*, and J_2^*, this formulation and ours are the same. To see this, consider any $h \in C^1$, for which $h(t)=0$ for $t \geq T$, for some $T > t_1$, and $h(t_1) \neq 0$. Define $\epsilon = \lambda h(t_1)$, for $\lambda \in \mathbb{R}$. We clearly have

$$J_2^*(x(t_1)+\epsilon,t_1)-J_2^*(x(t_1),t_1) \geq \int_{t_1}^{T} L^2(x+\lambda h,\dot{x}+\lambda\dot{h},t,t_1)-\int_{t_1}^{T} L^2(x,\dot{x},t,t_1).$$

Thus $\frac{J_2^*(x(t_1)+\epsilon,t_1)-J_2^*(x(t_1),t_1)}{\epsilon} \geq \int_{t_1}^{T} \frac{L^2(x+\lambda h,\dot{x}+\lambda\dot{h},t,t_1)-L^2(x,\dot{x},t,t_1)}{\lambda h(t_1)}$, if $\epsilon > 0$. For $\epsilon \downarrow 0$, by (2.2) (by a sequence of λ suitable with $\epsilon \downarrow 0$),

$$\lim_{\lambda\to 0}\int_{t_1}^{T} \frac{L^2(x+\lambda h,\dot{x}+\lambda\dot{h},t,t_1)-L^2(x,\dot{x},t,t_1)}{\lambda h(t_1)} = \int_{t_1}^{T} \frac{L_x^2(x,\dot{x},t,t_1)h(t)+L_{\dot{x}}^2(x,\dot{x},t,t_1)\dot{h}(t)}{h(t_1)} = 0.$$

Thus, $\frac{\partial J_2^*(x(t_1),t_1)}{\partial x(t_1)} = \lim_{\epsilon\downarrow 0}\frac{J_2^*(x(t_1)+\epsilon,t_1)-J_2^*(x(t_1),t_1)}{\epsilon} \geq 0$. By considering the limit $\epsilon \uparrow 0$, we obtain the reverse inequality, so that we have $\frac{\partial J_2^*(x(t_1),t_1)}{\partial x(t_1)} = -L_{\dot{x}}^2 \mid_{t_1+}$. A similar argument for the first period yields $\frac{\partial J_1^*(x(t_1),t_1)}{\partial x(t_1)} = L_{\dot{x}}^1 \mid_{t_1+}$. Thus we obtain by (2.13)

$$L_{\dot{x}}^1 \mid_{t_1+} = L_{\dot{x}}^2 \mid_{t_1+} .$$

But this is the result of a much more general result, namely the equivalence of the Pontryagin's conditions, and the Euler-Lagrange equation. By rewriting the Hamiltonians above, as $H^i(x,p,t,t_1) = -L^i(x,\dot{x},t,t_1)e^{-\rho t}+p(t)\dot{x}$, for the two periods, excluding the control from the definitions, we are able to state the following. Under the conditions that L is C^2, and $L_{\dot{x}\dot{x}}$ is invertible, say $L_{\dot{x}\dot{x}} < 0$, a solution of Euler-Lagrange equation is a solution of the corresponding Hamiltonian system, i.e. the equation system (2.12), and vice versa (Buttazzo, et.all 1998, Proposition 1.34, p.38). Moreover $L_{\dot{x}}(x, \dot{x}, t, t_1)e^{-\rho t} = p(t)$, at any t, which establishes the equality we have claimed.

The stated matching condition for an interior switch, at Tomiyama and Rossana (1989) is:

$$\left[H^2 \mid_{t_1+}\right] - \left[H^1 \mid_{t_1-}\right] - \int_{t_0}^{t_1} \frac{\partial H^1}{\partial t_1} dt - \int_{t_1}^{t_f} \frac{\partial H^2}{\partial t_1} dt = 0. \tag{2.14}$$

When the switching instant to not appear explicitly in the integrands or constraints, this condition reduces to $\left[H^2 \mid_{t_1+}\right] = \left[H^1 \mid_{t_1-}\right]$. This is the second Weierstrass-Erdman corner condition, and this is the formulation that we find in Makris(2001), as he does not consider explicit dependence on the switching instant.

The equivalence of this formulation to ours should be clear from the discussion so far, as we have

$$\begin{aligned} H^2 \mid_{t_1+} &= [\dot{x}L^2_{\dot{x}} - L^2]_{t_1+} e^{-\rho t_1} \\ H^1 \mid_{t_1-} &= [\dot{x}L^1_{\dot{x}} - L^1]_{t_1-} e^{-\rho t_1} \end{aligned}$$

$$\frac{\partial L^i(x, \dot{x}, t, t_1)}{\partial t_1} + \frac{\partial H^i(x, L_{\dot{x}}, t, t_1)}{\partial t_1} = 0, \text{ for } i \in \{1, 2\}.$$

CHAPTER 3

APPLICATION

In this part we will solve an adoption problem with expanding technology frontier. As advancement of technology may be regarded as a continuous process while adoption of it is a discrete process, the exercise below will be legitimate in its approach to the adoption issue. So it will help in understanding the dynamics of adoption. Yet, the exercise below should be treated as a complement to the studies Boucekkine et.all (2004), and Karaşahin (2006), as adoption process is rather complicated with determinants like learning and maintenance, effects of which are studied by these authors.

3.1 The Model

The model will be the following one:

$$\max_{k(t),t_1} \int_0^\infty \ln(c(t))e^{-\rho t}dt$$

subject to

$$\begin{aligned}
\dot{k}(t) &= q(0)(a_1 k(t) - c(t)), \text{ for } t < t_1 \\
\dot{k}(t) &= q(t_1)(a_2 k(t) - c(t)), \text{: for } t > t_1, \\
k(0) &= k_0, \; k(t) \geq 0, \; c(t) \geq 0
\end{aligned}$$

This is a representative agent model, with the intertemporal utility function $\int_0^\infty \ln(c(t))e^{-\rho t}dt$. Here $\rho > 0$ represents the discount factor, k, and c

represents capital stock, and consumption, respectively. The relations,

$$\begin{aligned} y(t) &= a_i k(t) = c(t) + I(t) \\ \dot{k}(t) &= q(t)\ I(t), \end{aligned}$$

imply the stated form of the problem. The first of these is resource constraint, for $i = 1, 2$, refering to the periods, and $a_i k(t)$ to the production function, and the second one gives the evolution of capital stock, where I is investment, $q(t)$ is the level of technology. Note that, we have no depreciation, so by $I \geq 0$, we will always have $\dot{k}(t) \geq 0$.The initial capital stock k_0 will assumed to be positive. By the production function, marginal productivity of the capital at the first period is given by a_1, and at the second period it is given by a_2. We assume that $a_2 < a_1$. The reasoning behind this assumption is provided in the introduction. $q(t)$ is assumed to be $1 + \gamma t$, i.e. we assume a linearly expanding frontier, so that when switching is realized, the adopted level of technology will be $q(t_1) = 1 + \gamma t_1$, while before switching it is $q(0) = 1$, without loss of generality.

The basic interpretation of this setup is provided in the introduction. Here the only change, with respect to the basic problem discussed at the introduction is explicit specification of the level of technology at each instant. Interpreting this setup further as a technology adoption problem of a developing country may also help in understanding the model. A developing country importing technology from abroad would solve this problem to find the optimal timing of adoption under a maximum of one switching allowed.

Before analyzing the solution to the model we will check the validity of the assumptions made in the previous part. The existence of solution follows from the existence of solution for standard optimal growth models (see d'Albis et.all 2004). This establishes existence of a continuous solution for both periods of the problem, separately. So existence problem reduces to the existence of solution to the problem in which the function

$$\Psi(t_1, x(t_1)) = \sup_{y,\, y(t_1)=x(t_1)} \left[\int_{t_0}^{t_1} L^1(y, \dot{y}, t, t_1) e^{-\rho t} dt + \int_{t_1}^{t_f} L^2(y, \dot{y}, t, t_1) e^{-\rho t} dt \right]$$

is maximized (right hand side attains its supremum). We impose continuity to the solution at the switching instant by setting $y(t_1) = x(t_1)$ for all y in the solution set. So let $\Psi(t_n, x(t_n)) \rightarrow \sup_{t_1, x(t_1)} \Psi(t_1, x(t_1))$. If $t_n \rightarrow \infty$, then existence is no problem since problem reduces to a one stage problem. Otherwise t_n is bounded, from which it follows that $x(t_n)$ is also bounded. So by the continuity of Ψ, we conclude that Ψ attains its supremum.

Now, by the continuity of the solution k, and by the constraints it is obvious that $\dot{k}(t)$ is essentially bounded. From this follows, by a version of the fundamental theorem of calculus (see Gordon (1994), Theorem 6.27, p.104) that for a, b finite, $k(b) = k(a) + \int_a^b \dot{k}(t)dt$, i.e. $k(t)$ is absolutely continuous, i.e. $k(t) \in W^{1,1}(loc)$. So, A4 is satisfied.

Given the continuity of $k(t)$, $\dot{k}(t) \geq 0$, and $k_0 > 0$, the assumption that $k(t) > 0$, uniformly on bounded intervals, is trivially satisfied. But the remaining part of the assumption A3 is not so trivial. It is equivalent here to the argument that $c(t)$ be uniformly above zero, on bounded intervals. This is established for standard optimal growth models (see Le Van et.all, 2007), but when there is a switch the issue becomes rather complicated. So we will take it as given in the present study.

Continuity and the differentiability of the functions involved in the problem are obvious (for ℓn we need to restrict the domain of the function, so that the restricted domain includes the solution, but the interiority assumption above shows that this is possible). It remains to consider the assumptions, A5, A6 and A7, but we will be able to check them right before we write the matching condition.

3.2 Solution

By Proposition 2, on each period $c(t)$ and $k(t)$ are differentiable. Having this in mind, by (2.1) for the second period we have

$$\frac{\dot{c}(t)}{\alpha\,(c(t))^2} + \frac{1}{\alpha\, c(t)}\,\rho = \frac{a_2}{c(t)}.$$

From this we find that,

$$c(t) = c(t_1+)\; e^{(\rho - a_2\alpha)t_1}\; e^{(a_2\alpha-\rho)t}.$$

Using this we obtain the first order linear differential equation, for $A = c(t_1)\; e^{(\rho-a_2\alpha)t_1}$,

$$\dot{k}(t) - a_2\,\alpha\, k(t) + A\,\alpha\, e^{(a_2\alpha-\rho)t} = 0,$$

solution of which is:

$$k(t) = -A\,\alpha\, e^{a_2\alpha\, t}\left[-\frac{e^{-\rho t}}{\rho} - \frac{k(t_1)\; e^{-a_2\alpha\, t_1}}{A\,\alpha} + \frac{e^{-\rho t_1}}{\rho}\right] \tag{3.1}$$

Following Boucekkine et.all (2004), the necessary transversality condition is :

$$\lim_{t\to\infty}\left(\frac{\partial L}{\partial \dot{k}} k(t) e^{-\rho t}\right) = 0.$$

This limit is $\dfrac{e^{-\rho t_1}}{\rho} - \dfrac{k(t_1)\; e^{-a_2\alpha\, t_1}}{A\,\alpha}$, so we conclude that:

$$c(t_1+) = \frac{\rho}{\alpha}\, k(t_1) \tag{3.2}$$

Again utilizing (2.1), this time for the first period we find that:

$$c(t) = c(0)\; e^{(a_1-\rho)t} \tag{3.3}$$

Writing the equation for capital, we find:

$$k(t) = -c(0)e^{a_1 t}\Big[-\frac{e^{-\rho t}}{\rho} + \frac{1}{\rho} - \frac{k(0)}{c(0)}\Big] \tag{3.4}$$

The corollary 2 above states that $\frac{\partial L}{\partial \dot{k}}$ is continuous at t_1. Since

$$\lim_{t \to t_1^+} \frac{\partial L}{\partial \dot{k}} \;\; = \frac{-1}{\rho\, k(t_1)}$$

and,

$$\lim_{t \to t_1^-} \frac{\partial L}{\partial \dot{k}} = \lim_{t \to t_1^+} \frac{\partial L}{\partial \dot{k}},$$

by equality of these:

$$c(0) = \rho\; k(t_1)\; e^{(\rho - a_1)t_1}. \tag{3.5}$$

We also have continuity of $k(t)$ at t_1. Writing (3.4) at t_1 :

$$\begin{aligned} k(t_1) \;\; &= -c(0)e^{a_1 t_1}\Big[-\frac{e^{-\rho t_1}}{\rho} + \frac{1}{\rho} - \frac{k(0)}{c(0)}\Big] \\ &= -\rho\; k(t_1)\; e^{(\rho-a_1)t_1} e^{a_1 t_1}\Big[-\frac{e^{-\rho t_1}}{\rho} + \frac{1}{\rho} - \frac{k(0)}{\rho\; k(t_1)\; e^{(\rho-a_1)t_1}}\Big] \\ &= k(t_1) - k(t_1)\; e^{\rho t_1} + k(0)\, e^{a_1 t_1} \end{aligned}$$

From this follows that:

$$k(t_1) = k(0)\; e^{(a_1 - \rho)t_1} \tag{3.6}$$

Thus we have the solution of the problem in terms of $k(0)$, and t_1. We summarize this below:

$$k(t) = k_0\; e^{(a_1-\rho)t},\; 0 < t \leq t_1 \tag{3.7}$$

$$c(t) = \rho\, k_0\; e^{(a_1-\rho)t},\; 0 < t < t_1 \tag{3.8}$$

$$k(t) = k_0\, e^{(a_1 - a_2\alpha)t_1}\ e^{(a_2\alpha - \rho)t},\ t_1 \leq t < \infty \tag{3.9}$$

$$c(t) = \frac{\rho}{\alpha}\, k_0\ e^{(a_1 - a_2\alpha)t_1}\ e^{(a_2\alpha - \rho)t},\ t_1 < t < \infty \tag{3.10}$$

Now it comes to utilize the matching condition, (3.12). First we have to check that A5, A6 and A7 are satisfied. Note that f^i in A5 is consumption. As the first period consumption do not depend on t_1 there is nothing to check. For the second period, as the consumption path is exponential, a perturbation of t_1 cannot make consumption negative. So A5 is satisfied. For A6, again we need to check only the second period as t_1 do not occur in the first period consumption. The second period L_{t_1} is $a_2 \frac{d\alpha}{dt_1} t e^{-\rho t}$, which is clearly integrable. So, A6 is also satisfied. For A7, we have $\ddot{k}(t) = k_0(-a_2\alpha + \rho)^2 e^{(a_1 - a_2\alpha)t_1 + (a_2\alpha - \rho)t}$. This function is locally integrable, obviously. Given these, we can proceed to characterize the switching instant.

To write the matching condition, we have:

$$\left[\dot{x}L^1_{\dot{x}} - L^1\right]_{t_1-} e^{-\rho t_1} = -\frac{\left(\rho\left(-1 + \ln\left(k_0 \rho e^{t_1(-\rho + a_1)}\right)\right) + a_1\right) e^{-\rho t_1}}{\rho}$$

$$\left[\dot{x}L^2_{\dot{x}} - L^2\right]_{t_1+} e^{-\rho t_1} = \frac{\left(\rho - \rho \ln\left(\frac{k_0 \rho e^{t_1(-\rho + a_1)}}{1 + \gamma t_1}\right) - a_2(1 + \gamma t_1)\right) e^{-\rho t_1}}{\rho}$$

$$\int_{t_1}^{t_f} L^2_{t_1} e^{-\rho t} dt = \frac{\left(-\gamma\rho + (1 + \gamma t_1)\left(\rho a_1 - a_2(\rho + \gamma(-1 + \gamma t_1)\ )\right)\right) e^{-\rho t_1}}{(1 + \gamma t_1)\rho^2}$$

Also by, $\int_0^{t_1} L^1_{t_1} e^{-\rho t} dt = 0$, the matching equation is:

$$\frac{e^{-\rho t_1}\left\{\rho\ \left[\gamma-(1+\gamma t_1)\rho\ln(1+\gamma t_1)\right]+(1+\gamma t_1)\left[-2\rho a_1+a_2(2\rho+\gamma(2\rho t_1-1))\right]\right\}}{(1+\gamma t_1)\rho^2} \tag{3.11}$$

So the necessary condition for an interior switch is:

$$\rho\ \left[\gamma-(1+\gamma t_1)\rho\ln(1+\gamma t_1)\right]+(1+\gamma t_1)\left[-2\rho a_1+a_2(2\rho+\gamma(2\rho t_1-1))\right]=0. \tag{3.12}$$

After some manipulation, and defining $s = 1 + \gamma t_1$, we reduce this to:

$$\rho\,\gamma + 2\rho a_2 s^2 = \rho^2 s\ln s + s(2\rho(a_1-a_2) + a_2\gamma + 2\rho a_2). \tag{3.13}$$

As a first step in the interpretation of this equation note that this equation does not depend on the initial value of the capital stock. This is rather natural in our setup, since marginal productivity of capital does not depend on the amount of capital stock as we assume Ak type production technology. For a Cobb-Douglas type technology switching instant would depend on the initial capital stock.

To simplify the interpretation of (3.13), we will assume that

$$\rho\gamma < 2\rho(a_1-a_2) + a_2\gamma.$$

In this way we ensure that left hand side of (3.13) has a lower value than right hand side of (3.13) at $t_1 = 0$. Now, the derivative with respect to s at the left hand side of (3.13) is $4\rho a_2 s$, while the right hand side derivative is $\rho^2(\ln s + 1) + 2\rho(a_1 - a_2) + a_2\gamma + 2\rho a_2$. Since for large s the left hand side derivative will be higher than that of the right hand side, there is a unique solution $t_1 > 0$ to (3.13), under this assumption. Hence excluding the possibility of a corner solution, we may continue to analyze the matching condition. But before to that we note that, above inequality has an important interpretation that highlights the trade-off at the center of the problem. We have explained the costs and advantages of switching at the introduction. So if at t_1 values close to zero, the costs of switching is less than the advantage gained by switching to higher technology, and if this difference decreases up to a point, so that costs and benefits outweigh each other at that point, than this point should be the switching point. Indeed, the above inequality and the discussion following it ensures this.

The matching condition does not have an algebraic solution. So we will proceed with the examination of elasticities of the parameters in (3.13), and do some numerical calculations. To understand the effect of a_2, second period marginal productivity of capital, on the switching instant, we will calculate the switching instants for the following set of parameters: $\rho = 0.04$, $\gamma = 0.02$, $a_1 = 1$ (for a discussion of this particular choice of parameters, see Sağlam (2002)) as

a_2	t_1
0.8	25.1
0.7	34.25
0.6	46.5

The interpretation is that, lower value of marginal productivity after adoption delays the adoption. This is reasonable since lower marginal productivity after adoption means that the costs to switching is higher. So this should be compensated by a higher gain in technological jump, so by waiting a higher technology to adopt. This is more clear if we consider the derivative with respect to a_2 of (3.12), as this derivative, $-\gamma(1+\gamma t_1) + 2(1+\gamma t_1)^2 \rho$, is positive whenever $\rho \geq \frac{\gamma}{2}$.

ρ	t_1
0.03	29.12
0.04	25.1
0.06	21.05

Higher discount rates should fasten the adoption. Higher discounting implies an urgency in covering the costs resulting from the delay in adoption. In fact the costs from switching decreases at a particular instant with higher discount rates with respect to the costs with a lower discount rates. This is what we see in the above table constructed with the parameter set $\gamma = 0.02$, $a_1 = 1$, $a_2 = 0.8$. Again, by looking at the derivative of (3.13) with respect to ρ, $-(2(a_1 - a_2) + 2a_2)s + 2a_2 s^2 + \gamma - 2s\rho \ln(s)$, we see this effect (we look at the derivative with respect to s as it is easier to interpret).

Our final consideration will be about the effect of the pace of technology on adoption. Our findings are again rather intuitive. For the parameter set, $\rho = 0.04$, $a_1 = 1$, $a_2 = 0.8$, we obtain the following:

γ	t_1
0.02	25.1
0.06	16.64
0.1	14.98

Higher pace of technology implies fastening of adoption. As high technology comes early, the loss due to the drop in marginal productivity of capital after adoption becomes tolerable in a shorter run. Derivative of (3.12) with respect to γ is $\rho - t_1\rho(2a+\rho) + a_2(-1-2t_1\gamma+4t_1(1+t_1\gamma)\rho) - t_1\rho^2 \ln[1+t_1\gamma]$. Given the assumption of lower discount values above, this derivative supports our numerical analysis. We should also note the following numerical result: $0.02 \times 25.1 = 0.5 < 0.06 \times 16.64 = 0.99 < 0.1 \times 14.98 = 1.498$. This implies that, not only adoption gets earlier as technology increases faster, but also adopted level of technology gets higher.

CHAPTER 4

CONCLUSION

In this study we have formulated necessary conditions for a class of infinite horizon multi-stage optimization problems. We compared these with the literature, and applied to an economic problem. Meanwhile we extended the already established results, in some directions. In particular we have formulated the necessary conditions for multi-stage problems depending explicitly to the switching instant, in infinite horizon. These are done by treating the problem as an ordinary problem in calculus of variations, and attacking it with the standard tools in calculus of variations, together with basic properties of Sobolev spaces.

The advantages of our approach first rests on the fact that we never refer to a value function. Hence we avoid the strict assumption that the value function be differentiable. Second, we impose no regularity on the derivative of the state path except for a boundedness assumption. Yet, we are able to cover the most general problem in the literature, necessary conditions of optimality of which is never provided in the literature. This is also partially true when we consider the so-called hybrid optimal control literature in engineering, since the problem here do not exclude the explicit dependence to the switching instant. We say "partially" since hybrid optimal control literature includes results for nonsmooth problems, which we do not cover here.

As an application of these theoretical remarks, we have analyzed a technological adoption problem with a linearly expanding technological frontier.

We have demonstrated, mainly that, under some certain assumptions, an increase in the speed of technology induces faster adoption, and adoption of higher technologies.

Yet, this application does not reflect the strength of our theoretical results, as we allow for just one switch in infinite horizon. Indeed, higher number of switches, possibly number of switches determined endogenously, would also be more sensible from an economic point of view. An important extension would also be abandonment of the restriction that the new technology after switching is the highest possible technology available at that instant[1]. In fact, in reality it is rare that the highest possible technology is adopted when it is to adopt due to the higher price of the new technology. While it is really a hard task to handle the case of endogenous number of switches, as this would be a genuine combination of discrete and continuous optimization, the last extension can be handled within our framework by introducing a new parameter to the functional maximized and treating it as the switching instant is treated.

We have no result in this study on uniqueness and sufficiency. While in application part we ensure these easily, establishing general results in these requires some convexity like assumptions together with an evaluation of the second variation of the problem. So we also note these for future work.

[1]This important extension is suggested by Semih Koray.

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